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COMPACT COVARIANCE OPERATORS[1]

by

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ABSTRACT

Let B be a real separable Banach space and R: $B^* \to B$ a covariance operator. All representations of R in the form $\sum e_n \otimes e_n$, $\{e_n, n \geq 1\} \subset B$, are characterized. Necessary and sufficient conditions for R to be compact are obtained, including a generalization of Mercer's theorem. An application to characteristic functions is given.

KEY WORDS AND PHRASES: Covariance operators, compact operators, probability in Banach spaces.

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1. Introduction

The study of covariance operators is a major component in the theory of probability measures on Banach spaces ~~[10], [2], [11]~~. The covariance operator of a strong second-order measure is always compact [2]; however, the covariance operator of a weak second-order measure need not be compact. In this paper we first characterize series representations of covariance operators, and then give a set of necessary and sufficient conditions for a covariance operator to be compact. The classical Mercer's theorem [7] can be obtained as an immediate corollary. These results are then applied to extend a result of Prohorov and Sazanov [6] on relative compactness of probability measures from Hilbert space to Banach space.

2. Definitions and Notation

B is a real separable Banach space with norm $||\cdot||$ and topological dual B*. A linear operator $R: B^* \to B$ is a covariance operator if R is symmetric and non-negative: $\langle Ru,v\rangle = \langle u,Rv\rangle$ and $\langle Ru,u\rangle \geq 0$, for all u,v in B*. A probability measure μ on the Borel σ-field of B is said to be weak second-order if $\int_B \langle x,u\rangle^2 d\mu(x) < \infty$, for all u in B*; μ is strong second-order if $\int_B ||x||^2 d\mu(x) < \infty$. Every weak second order measure μ has a mean element m in B and a covariance operator $R: B^* \to B$ [9], [10], defined by

$$\langle m,v\rangle = \int_B \langle x,v\rangle d\mu(x)$$
$$\langle Ru,v\rangle = \int_B \langle x-m,u\rangle\langle x-m,v\rangle d\mu(x) \quad ,$$

for all u,v in B*. Strong second-order measures have compact covariances; the strong second order property is not necessary in order that μ have compact covariance.

For a covariance operator $R: B^* \to B$ it is well known [8], [1], that there exists a separable Hilbert space $H \subset B$ such that the natural injection

$j: H \to B$ is continuous and $R = jj^*$. H is the RKHS of R and is the completion of range (R) with respect to the inner product $<\cdot,\cdot>_H$ defined by $<Ru,Rv>_H = <Ru,v>$.

I_H will denote the identity on H. For u,v in B*, z in B (resp. in H), $(u \otimes v)(z) = <v,z>u$ (resp., $<v,z>_H u$). If T is any map $r(T) \equiv \text{range}(T)$. τ_C is the linear topology on B* determined by a neighborhood base at zero of the form $V_{C,\varepsilon}(0) = \{f \in B^*: \sup_{x \in C} <f,x>^2 < \varepsilon\}$ for all $\varepsilon > 0$ and all compact sets $C \subset B$ (τ_C is the topology of uniform convergence on compact sets). For a given covariance operator $R: B^* \to B$, q_R is the real-valued quadratic functional on B* defined by $q_R f = <Rf,f>$. The notation $R = \sum_n e_n \otimes e_n$ for $\{e_n, n \geq 1\} \subset B$ means that the sequence $(\sum_1^N e_n \otimes e_n)$ converges to R in the strong operator topology: $\sum_1^N <e_n,f>e_n \to Rf$ in the norm topology of B, for all f in B*. $I_H = \sum_n e_n \otimes e_n$ has a similar interpretation. If $\{u_n, n \geq 1\}$ is any orthonormal basis for H, then $R = \sum ju_n \otimes ju_n$, [9]. K_R will denote the unit ball in H.

If μ is a probability measure on the Borel σ-field of B, its characteristic functional $\hat{\mu}$ is defined as

$$\hat{\mu}(x) = \int_B e^{i<x,y>} d\mu(y) \ , \quad \text{for } x \text{ in } B^* \ .$$

3. Representation of Covariance Operators.

In this section, R is an arbitrary covariance operator.

Theorem 1. $R = \sum_n e_n \otimes e_n$ if and only if $e_n = jv_n$, $v_n \in H$ for $n \geq 1$, and $I_H = \sum_n v_n \otimes v_n$.

Proof. It suffices to show that the stated conditions are necessary for $R = \sum_n e_n \otimes e_n$. Suppose $R = \sum_n e_n \otimes e_n$, and fix e_k. Let $P_k = e_k \otimes e_k$. To show $e_k \in \text{range}(j)$, let (as in [3]) $D\colon r(j^*) \to B$ be defined by $Dj^*f = P_k f$. Then $||Dj^*f||^2 = ||P_k f||^2 = ||e_k||^2 \langle e_k, f\rangle^2 \leq ||e_k||^2 \sum_n \langle f, e_n\rangle^2 = ||e_k||^2 \langle Rf, f\rangle = ||e_k||^2\, ||j^*f||_H^2$. Thus D can be extended to a continuous linear map from $\overline{r(j^*)} = H$ into B. From its definition, $Dj^* = P_k$, so $P_k = jD^*$ and thus $e_k \in \text{range}(j)$.

To see that $I_H = \sum v_n \otimes v_n$, where $jv_n = e_n$, $n \geq 1$, define $Q_N = \sum_1^N v_n \otimes v_n$. $Q_N = Q_N^*$ and $Q_N \geq 0$, so $Q_N^{\frac{1}{2}}$ exists. $||Q_N^{\frac{1}{2}} j^*f||_H^2 = \sum_1^N \langle f, e_n\rangle^2 \uparrow ||j^*f||_H^2$, so that $||Q_N^{\frac{1}{2}}|| \leq 1$ and $||Q_N^{\frac{1}{2}} x||_H \to ||x||_H$ for all x in $r(j^*)$. Thus,

$$||\textstyle\sum_1^N (v_n \otimes v_n) j^*f - j^*f||_H^2 = ||Q_N j^*f - j^*f||_H^2$$

$$\leq -||Q_N^{\frac{1}{2}} j^*f||_H^2 + ||j^*f||_H^2 ,$$

which converges to zero as $N \to \infty$ for any fixed f in B^*. Thus, $\sum v_n \otimes v_n = I_H$ on $r(j^*)$, and the result follows by $\overline{r(j^*)} = H$. □

Remark. Suppose E is a locally convex topological vector space, $R\colon E' \to E$ is a covariance operator, and $R = jj^*$, where $j\colon H \to E$ is the injection and H is the RKHS of R. R will have such a representation, for example, if E is separable and quasi-complete [8]. In this case, it is easily shown that *Theorem 1 holds without modification.*

The representation $I_H = \sum v_n \otimes v_n$ does not require that $\{v_n, n \geq 1\}$ be a CONS in H; however, sufficient conditions for $\{v_n, n \geq 1\}$ to be a CONS in H can be given.

Proposition 1. Suppose $I_H = \sum v_n \otimes v_n$; the following are equivalent:

(1) $||v_k||_H = 1$

(2) $v_k \notin \overline{\text{sp}\{v_n, n \neq k\}}$

(3) $v_k \perp \overline{\text{sp}\{v_n, n \neq k\}}$.

If any of the above conditions hold for all $k \geq 1$, then $\{v_n, n \geq 1\}$ is a CONS for H.

4. Compact Covariance Operators.

Theorem 2. Suppose $R = \sum e_n \otimes e_n$, $\{e_n, n \geq 1\} \subset B$. Let $\{v_n, n\geq 1\} \subset H$ be such that $e_n = jv_n$, $n \geq 1$. The following are equivalent:

(1) R is compact;

(2) j is compact;

(3) $j[K_R]$ is compact in B;

(4) the series $\sum v_n \otimes jv_n$ converges uniformly in H on bounded subsets of B^*;

(5) $(\sum_1^N e_n \otimes e_n)$ converges to R uniformly in B on bounded subsets of B^*;

(6) q_R is w^*-continuous on bounded subsets of B^*;

(7) q_R is τ_C-continuous.

Proof. (1) => (2). Suppose $f_\alpha \to f$ in the w^* topology of B^*, where $\|f_\alpha\| \leq k$ for all α. Then $\|j^*f_\alpha - j^*f\|_H^2 = \langle R(f_\alpha - f), (f_\alpha - f)\rangle \leq 2k\|R(f_\alpha - f)\|_B$; since R is compact, $j^*f_\alpha \to j^*f$ in H [4, p. 486] and thus j is compact.

(2) => (3). j compact implies $j[K_R]$ is relatively compact in B. Since K_R is weakly compact in H and j is weakly continuous, $j[K_R]$ is weakly compact in B, and thus closed.

(3) => (2) by definition.

(2) => (4). By Theorem 1, $\sum_n v_n \otimes v_n = I_H$. Set $Q_N = \sum_1^N v_n \otimes v_n$. If $A \subset B^*$ is bounded, then $\overline{j^*[A]}$ is compact; by Dini's theorem $\|Q_N^{\frac{1}{2}}x\|_H \uparrow \|x\|_H$ uniformly on $j^*[A]$. Hence $\|(Q_N - I)j^*x\|_H^2 \leq \|j^*x\|_H^2 - \|Q_N^{\frac{1}{2}}j^*x\|_H^2 \to 0$ uniformly on A.

(4) => (5), since j is continuous.

(5) => (1), since R is the uniform limit of compact operators.

(2) <=>(6). Follows from the fact that j is compact if and only if $j^*f_\alpha \to 0$ in the norm topology of H for all bounded generalized sequences (f_α) in B^* which are w^* convergent to zero [4, p. 486], and $q_R(f_\alpha) = \|j^*f_\alpha\|_H^2$.

(1) => (7). Suppose R is compact. Writing $C = j[K_R]$, C is compact in B. $q_R(f) = \langle Rf,f\rangle = ||j^*f||_H^2 = \sup_{x\in K}\langle j^*f,x\rangle_H^2 = \sup_{x\in C}\langle f,x\rangle^2$. Thus q_R is τ_C-continuous at zero. τ_C-continuity of q_R follows from $q_R(f_\alpha) = q_R(f_\alpha - f) - q_R(f) + 2\langle Rf,f_\alpha\rangle$.

(7) => (1). Suppose q_R is τ_C-continuous. Using (6), R is compact if q_R is w* continuous at 0 on bounded subsets of B*. But B is separable so that the w* topology on bounded subsets of B* is metrizable and it suffices to consider sequences. Suppose $f_n \overset{w^*}{\to} 0$ and $||f_n|| \le k$. Let L be an arbitrary compact subset of B. Since $\{f_n\}$ is bounded in B* the f_n are equicontinuous and uniformly bounded as continuous functions on L. Thus, by the Arzela-Ascoli Theorem [4, p. 266] $\{f_n\}$ is relatively compact as a subset of $C^{\mathbb{R}}(L)$. Thus since $f_n \overset{w^*}{\to} 0$, f_n converges to 0 uniformly on L. Therefore $f_n \overset{\tau_C}{\to} 0$ and $q_R(f_n) \to 0$. This completes the proof of Theorem 2. □

<u>Remarks</u>. (1) Suppose $r: [0,1]\times[0,1] \to \mathbb{R}$ is continuous, symmetric and positive definite. For fixed $t\in[0,1]$, let $\pi_t(x) = x_t$ for x in C[0,1]; $||\pi_t|| = 1$. A compact covariance operator R: C*[0,1] → C[0,1] is defined by $[R\mu](t) = \int_0^1 r(t,s)d\mu(s)$ for any μ in C*[0,1] (by Arzela-Ascoli Theorem). Thus for $s,t\in[0,1]$, $\langle R\pi_t,\pi_s\rangle = r(t,s)$. The integral operator in $L_2[0,1]$, corresponding to the kernel r, has continuous orthonormal eigenvectors $\{y_n, n\ge 1\}$ and associated non-zero eigenvalues $\{\lambda_n, n\ge 1\}$; it is well known that $\{\lambda_n^{1/2} y_n, n\ge 1\}$ is a CONS in the RKHS H of R. Thus, from Theorem 2, $\sum_{n=1}^N \lambda_n y_n(t) y_n(s)$ converges uniformly to r(t,s) for all t,s in [0,1]. This is the classical Mercer's Theorem [7, pp. 245-246].

(2) The fact that the unit ball of H is compact in B when R is compact was proved by Kuelbs [5] under the assumption that R is the covariance of a strong second-order measure.

5. Characteristic Functionals

Let Λ denote a family of probability measures on B (separable Banach) and $\hat{\Lambda}$ the corresponding family of characteristic functionals.

Theorem 3. Let B be a separable Banach space. Then the following are equivalent:

a) There exists a topology τ on B^* such that for each family Λ of probability measures on B, $\hat{\Lambda}$ is equicontinuous in this topology if and only if Λ is relatively compact in the topology of weak convergence

b) B is finite dimensional.

Proof. As in the Hilbert space case (see [6, Lemma 2]) τ_c is the weakest topology on B^* such that relative compactness of $\Lambda \Rightarrow$ equicontinuity of $\hat{\Lambda}$. Suppose that (a) holds. Then $\tau_c \subset \tau$ and τ_c equicontinuity of $\hat{\Lambda}$ implies relative compactness of Λ. Now let $R: B^* \to B$ be any compact covariance operator. Let $\{e_n\}$ be a CONS in the RKHS of R. Define μ_k to be the zero mean Gaussian measure on B with covariance operator $\sum_1^k e_n \otimes e_n$. Then $\{\hat{\mu}_k\}$ is τ_c equicontinuous by Theorem 2 and $\{\mu_k\}$ is relatively compact. Therefore R is the covariance of a Gaussian probability measure on B and by [9, Theorem 11] B is finite dimensional. □

Theorem 3 extends a result of Prohorov and Sazonov [6] who proved it for Hilbert spaces.

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